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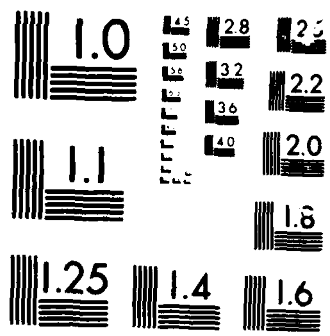

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THE CENTRAL LIMIT THEOREM AND POINCARÉ-TYPE INEQUALITIES

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ABSTRACT

We use Poincaré-type inequalities to prove the sufficiency and necessity of the Lindeberg condition in the central limit theorem.

AMS (MOS) Subject Classifications: Primary 60F05, 60E15; Secondary 26D10.

Key Words: Central limit theorem, Poincaré-type inequalities, characterization of the normal distribution

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SIGNIFICANCE AND EXPLANATION

The central limit theorem is a fundamental theorem in probability and statistics. It states that the probability distribution of the sum of a large number of small and mutually independent random numerical observations approaches a normal distribution as the number of observations increases. The Lindeberg condition is a condition for which the central limit theorem holds. It has been proved to be both necessary and sufficient.

A Poincaré-type inequality is an inequality which relates the integral of the square of a function to the integral of the square of its derivative. In this report we give a new proof of the central limit theorem by using Poincaré-type inequalities to prove both the necessity and sufficiency of the Lindeberg condition.

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Borovkov and Utev (1984) defined the functional

$$U(X) = \sup_g \frac{\mathrm{Var}[g(X)]}{\sigma^2 E[g'(X)]^2}$$

for any random variable X with finite variance σ^2, where the supremum is taken over the class of absolutely continuous functions g such that $0 < \mathrm{Var}[g(X)] < \infty$. They proved that

(1) $U(X) \geq 1$ and if $U(X) = 1$ then X has a normal distribution.

Using this result they further proved that

(2) if $X_1, X_2, \ldots$ is a sequence of random variables such that $U(X_n) \to 1$ then the moment generating function of $(X_n - EX_n)/[\mathrm{Var}(X_n)]^{1/2}$ exists and converges to that of the standard normal random variable in a neighborhood of zero.

It is natural to ask if (1) can also be applied to prove the central limit theorem under the Lindeberg condition. This question was in fact raised by S. Kotani in private communication with the author and motivated the present work.

The existence of the moment generating function of X_n in (2) is due to the finiteness of $U(X_n)$ (see Borovkov and Utev, Theorem 2). Since the central limit theorem does not require such a strong condition, arguments different from those of Borovkov and Utev would have to be used. It turns out that a Poincaré-type inequality for sums of independent random variables proved along the same line as in Chen (1985, Section 2) is the key to this problem. Using this inequality we can prove not only the sufficiency of the

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Lindeberg condition but also its necessity.

In order to facilitate application, we restate (1) in a different form. Let $C_B^1(\mathbb{R})$ be the class of functions g such that g and g' are bounded and continuous.

PROPOSITION 1. Let X be a random variable with finite variance $\sigma^2 > 0$. If $\mathrm{Var}[g(X)] \le \sigma^2 E[g'(X)]^2$ for $g \in C_B^1(\mathbb{R})$, then X has a normal distribution.

To see that (1) and Proposition 1 are equivalent, we define

$$U_0(X) = \sup_{\substack{g \in C_0^\infty(\mathbb{R}) \\ \mathrm{Var}[g(X)]>0}} \frac{\mathrm{Var}[g(X)]}{\sigma^2 E[g'(X)]^2}$$

and

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where $C_0^\infty(\mathbb{R})$ is the class of C^∞ functions on $\mathbb{R}$ with compact support. Then we observe that by definition, $U(X) \ge 1$ and that by Theorem 2(i) of Borovkov and Utev, $U_0(X) = U_B(X) = U(X)$.

Since we are interested in both an application of Poincaré-type inequalities and a new proof of the central limit theorem, it is fitting to examine the arguments which lead to (1) and hence Proposition 1. There are three different proofs of (1). All begin with a variational argument. After that the first uses the method of moments (see Borovkov and Utev, Theorem 3). The second uses the characteristic function (see Chen and Lou (1985, Theorem 2.1 and Corollary 2.1). The third uses differential equations (see Chen and Lou, Lemmas 4.1 and 4.2).

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We now prove the sufficiency of the Lindeberg condition. Let $X_{n1},\ldots,X_{nr_n}$, $n \geq 1$, be a triangular array of row-wise independent random variables with zero means and finite variances $\sigma_{n1}^2,\ldots,\sigma_{nr_n}^2$ such that $\sum_{i=1}^{r_n} \sigma_{ni}^2 = 1$. Let $W_n = \sum_{i=1}^{r_n} X_{ni}$ and $W_n^{(i)} = W_n - X_{ni}$. By Theorem 2.1 of Chen (1985),

$$\mathrm{Var}[g(W_n)] \leq \sum_{i=1}^{r_n} E\Big[\mathrm{Var}^{W_n^{(i)}} g(W_n)\Big]$$

for $g \in C^1(\mathbb{R})$ such that $|g(x)| \leq C(1 + |x|)$ for some constant C. Now the right hand side of the inequality equals

$$\sum_{i=1}^{r_n} FE^{W_n^{(i)}}\big[g(W_n) - g(W_n^{(i)}) - E^{W_n^{(i)}}(g(W_n) - g(W_n^{(i)}))\big]^2$$

$$\leq \sum_{i=1}^{r_n} E[g(W_n) - g(W_n^{(i)})]^2$$

$$= \sum_{i=1}^{r_n} E\Big[\int_0^{X_{ni}} g'(W_n^{(i)} + t)dt\Big]^2$$

$$\leq \sum_{i=1}^{r_n} E\Big\{X_{ni}\int_0^{X_{ni}} [g'(W_n^{(i)} + t)]^2 dt\Big\}$$

$$= \sum_{i=1}^{r_n} E\Big\{X_{ni}\int_{-\infty}^{\infty} [I(X_{ni} > t > 0) - I(X_{ni} < t < 0)][g'(W_n^{(i)} + t)]^2 dt\Big\}$$

$$= \sum_{i=1}^{r_n} \int_{-\infty}^{\infty} E[g'(W_n^{(i)} + t)]^2 K_n^{(i)}(t)dt$$

where $K_n^{(i)}(t) = EX_i[I(X_{ni} > t > 0) - I(X_{ni} < t < 0)] \geq 0$. Define the probability measure ν_n on $\mathcal{B}(\mathbb{R}^2)$ by

$$\int_{\mathbb{R}^2} f d\nu_n = \sum_{i=1}^{r_n} \int_{-\infty}^{\infty} Ef(W_n^{(i)}, t)K_n^{(i)}(t)dt$$

for bounded and continuous functions f on $\mathbb{R}^2$. Also define $\psi : \mathbb{R}^2 \to \mathbb{R}$ by $\psi(x,y) = x+y$. Then, combining the above inequalities, we have the following Poincaré-type inequality for W_n : For $g \in C^1(\mathbb{R})$ such that $|g(x)| \leq C(1 + |x|)$ for some constant C,

$$\text{(3)} \qquad \operatorname{Var}[g(W_n)] \leq \int_{\mathbb{R}} (g')^2 d\nu_n \circ \psi^{-1} \quad .$$

Let $\nu_n^{(1)}(A) = \nu_n(A \times \mathbb{R})$ and $\nu_n^{(2)}(A) = \nu_n(\mathbb{R} \times A)$ for $A \in \mathcal{B}(\mathbb{R})$. Since

$$\begin{aligned} \nu_n^{(2)}(|y| > \varepsilon) &= \sum_{i=1}^{r_n} \int_{|t|>\varepsilon} K_n^{(i)}(t)dt \\ &= \sum_{i=1}^{r_n} E|X_{ni}|(|X_{ni}| - \varepsilon)^+ \\ &\leq \sum_{i=1}^{r_n} EX_{ni}^2 I(|X_{ni}| > \varepsilon) \quad , \end{aligned}$$

the Lindeberg condition implies that $\nu_n^{(2)} \Longrightarrow \varepsilon_0$, the Dirac measure at 0, as $n \to \infty$. Now $EW_n^{(i)2} \leq EW_n^2 = 1$ for each n and i implies that $\{\nu_n^{(1)}\}$ is tight and hence relatively compact. Let $\{\nu_{n'}^{(1)}\}$ be a weakly convergent subsequence and let $\nu_{n'}^{(1)} \Longrightarrow L(Z)$. Then $\nu_{n'} \Longrightarrow L\binom{Z}{0}$ and hence $\nu_{n'} \circ \psi^{-1} \Longrightarrow L(Z)$. It follows from (3) that for $g \in C_B^1(\mathbb{R})$,

$$\text{(4)} \qquad \operatorname{Var}[g(Z)] \leq E[g'(Z)]^2 \quad .$$

If we show that Z has a standard normal distribution, then $L(W_n) \Longrightarrow N(0,1)$ is proved. By Proposition 1 it remains to prove that $\operatorname{Var}(Z) = 1$. First we observe that by virtue of $EW_n^2 = 1$, Z is square integrable. Let $\varphi_a \in C_B^1(\mathbb{R})$ be increasing such that

$$\varphi_a(x) = \begin{cases} x & \text{if } |x| \leq a \\ a+1 & \text{if } x > a+2 \\ -a-1 & \text{if } x < -a-2 \quad . \end{cases}$$

By (3),

$$\operatorname{Var}[W_{n'} - \varphi_a(W_{n'})] \leq \int_{\mathbb{R}} (1-\varphi_a')^2 d\nu_{n'} \circ \psi^{-1} \quad .$$

Since $\nu_{n'} \circ \psi^{-1}$ converges weakly, $\{\nu_{n'} \circ \psi^{-1}\}$ is tight. Therefore for $\varepsilon > 0$,

$$\operatorname{Var}[W_{n'} - \varphi_a(W_{n'})] \leq \int_{\mathbb{R}} (1-\varphi_a')^2 d\nu_{n'} \circ \psi^{-1} \leq \varepsilon$$

for sufficiently large a. Now $(\mathrm{Var})^{1/2}$ is a seminorm and so

$$|1 - [\mathrm{Var}(\varphi_a(W_{n'}))]^{1/2}| = |[\mathrm{Var}(W_{n'})]^{1/2} - [\mathrm{Var}(\varphi_a(W_{n'}))]^{1/2}|$$

$$< [\mathrm{Var}(W_{n'} - \varphi_a(W_{n'}))]^{1/2}$$

$$< \varepsilon^{1/2} \quad .$$

By letting $n' \to \infty$ and then $a \to \infty$, we obtain

$$|1 - [\mathrm{Var}(Z)]^{1/2}| < \varepsilon^{1/2} \quad .$$

This implies that $\mathrm{Var}(Z) = 1$ and hence $L(W_n) \Longrightarrow N(0,1)$.

We now prove the necessity of the Lindeberg condition. First we need two *simple propositions*.

PROPOSITION 2. The Lindeberg condition holds if and only if $\nu_n^{(2)} \Longrightarrow \varepsilon_0$.

PROOF. The "only if" part has been proved above. The "if" part follows from the following inequalities:

$$\sum_{i=1}^{r_n} E|X_{ni}|(|X_{ni}| - \varepsilon)^+ > \sum_{i=1}^{r_n} E|X_{ni}|(|X_{ni}| - \varepsilon)I(|X_{ni}| > 2\varepsilon)$$

$$> \frac{1}{2}\sum_{i=1}^{r_n} EX_{ni}^2 I(|X_{ni}| > 2\varepsilon) \quad .$$

For the next proposition let $C_U^2(\mathbb{R})$ be the class of functions g on $\mathbb{R}$ such that g, g' and g'' are uniformly continuous, $|g(x)| < C(1 + |x|)$ for some constant C and g' and g'' are bounded.

PROPOSITION 3. Let Z and T be independent random variables such that Z has the normal distribution with mean 0 and variance $\sigma^2 > 0$. If $\mathrm{Var}[g(Z)] < \sigma^2 E[g'(Z+T)]^2$ for $g \in C_U^2(\mathbb{R})$, then $T = 0$ w.p. 1.

PROOF. By the variational argument used in Borovkov and Utev (1984) or Chen and Lou (1985), we have

$$EZh(Z) = \sigma^2 Eh'(Z+T)$$

for $h \in C_U^2(\mathbb{R})$. This equation also holds for $h(x) = x^3$ by approximating this function by functions of $C_U^2(\mathbb{R})$. So

$$3\sigma^4 = EZ^4 = 3\sigma^2 E(Z+T)^2 \quad .$$

The finiteness of $E(Z+T)^2$ and EZ^2 implies that of ET^2. By expanding $E(Z+T)^2$ we obtain $\sigma^2 = \sigma^2 + ET^2$ which implies that $ET^2 = 0$ and so $T = 0$ w.p. 1.

For the proof of the necessity of the Lindeberg condition we need the usual assumption that for every $\varepsilon > 0$, $\max_{1 \le i \le r_n} P(|X_{ni}| > \varepsilon) \to 0$ as $n \to \infty$. Let $g \in C_U^2(\mathbb{R})$. For every $\varepsilon > 0$, let $\delta > 0$ be such that $|(g'(x))^2 - (g'(y))^2| < \varepsilon$ for $|x-y| < \delta$. Then we have

$$\left| \sum_{i=1}^{r_n} \int_{-\infty}^{\infty} E\{(g'(W_n^{(i)} + t))^2 - (g'(W_n + t))^2\} K_n^{(i)}(t)dt \right|$$

$$\le C\{ \sum_{i=1}^{r_n} P(|X_{ni}| > \delta) \int_{-\infty}^{\infty} K_n^{(i)}(t)dt + \varepsilon \sum_{i=1}^{r_n} \int_{-\infty}^{\infty} K_n^{(i)}(t)dt\} \tag{5}$$

$$\le C[\max_{1 \le i \le r_n} P(|X_{ni}| > \delta) + \varepsilon]$$

for some constant C. Define the probability measure $\tilde{\nu}_n$ on $\mathcal{B}(\mathbb{R})$ by

$$\int_{\mathbb{R}} f d\tilde{\nu}_n = \sum_{i=1}^{r_n} \int_{-\infty}^{\infty} f(t) K_n^{(i)}(t)dt$$

for bounded and continuous functions f on $\mathbb{R}$. In view of (5), the inequality (3) can be written as

$$\mathrm{Var}[g(W_n)] \le \int_{-\infty}^{\infty} E[g'(W_n+t)]^2 \, \tilde{\nu}_n(dt) + o(1) \quad . \tag{6}$$

Suppose $\mathcal{L}(W_n) \Longrightarrow N(0,1)$. Let $\{\tilde{\nu}_{n'}\}$ be a subsequence of $\{\tilde{\nu}_n\}$ which converges vaguely to a subprobability measure $\tilde{\nu}$. Then for $g \in C_0^2(\mathbb{R})$, the class of C^2 functions on $\mathbb{R}$ with compact support, it is not difficult to show that

$$\int_{-\infty}^{\infty} E[g'(W_n + t)]^2 \, \tilde{\nu}_{n'}(dt) \to \int_{-\infty}^{\infty} E[g'(Z+T)]^2 \, \tilde{\nu}(dt)$$

where Z is a standard normal random variable. So by (6) we have

$$\text{(7)} \qquad \operatorname{Var}[g(Z)] \le \int_{-\infty}^{\infty} E[g'(Z+T)]^2 \, \tilde{\nu}(dt) \quad .$$

By approximating $C_U^2(\mathbb{R})$ by functions of $C_0^2(\mathbb{R})$, (7) holds for $g \in C_U^2(\mathbb{R})$. By letting $g(x) = x$, we get

$$1 \le \int_{-\infty}^{\infty} \tilde{\nu}(dt)$$

and so $\tilde{\nu}$ is a probability measure. By Proposition 3, $\tilde{\nu}$ must be ε_0. Hence $\tilde{\nu}_n \Longrightarrow \varepsilon_0$. But $\tilde{\nu}_n = \nu_n^{(2)}$. By Proposition 2, the Lindeberg condition holds. This completes the proof.

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